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2001 J. Phys.: Condens. Matter 13 L891

(http://iopscience.iop.org/0953-8984/13/44/102)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **13** (2001) L891–L897 PII: S0953-8984(01)27518-3

**LETTER TO THE EDITOR**

# Strongly interacting Luttinger liquid—exact solution of a generalized $t$–$J$ model in one dimension

**Igor N Karnaukhov**[1] **and Natan Andrei**[2]

[1] Institute of Metal Physics, Vernadsky Street 36, 03142 Kiev, Ukraine
[2] Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA

Received 2 August 2001
Published 19 October 2001
Online at stacks.iop.org/JPhysCM/13/L891

**Abstract**

Luttinger liquids are characterized by the critical exponent $\Theta$ of the momentum distribution around the Fermi momentum $k_F$. Typically $\Theta \leqslant 1/2$, signalling a singularity characterizing a residual Fermi surface. Results of photoemission experiments can be interpreted in terms *strongly interacting Luttinger liquids* with $\Theta \geqslant 1$ with the residual Fermi surface disappearing. We construct integrable models with such behaviour—models given by the SU($\nu$) $t$–$J$ interaction with a hard-core repulsive potential between electrons at distances less than or equal to $\Delta$. The models exhibit both weakly and strongly interacting Luttinger behaviours with $\Theta$ varying continuously in the range $0 \leqslant \Theta \leqslant \frac{1}{2}(1 + \Delta - 1/[\nu(1 + \Delta)])^2$ depending on the electron density. In the extreme high-density limit the model exhibits a Mott–Hubbard gap and reduces to an isotropic Heisenberg chain with a new spacing parameter $\Delta + 1$.

Exactly solvable models of strongly correlated systems have been intensively studied recently, with the aim of understanding the mechanisms underlying high-$T_c$ superconductivity. An example is the $t$–$J$ model [1] which has dominant superconducting correlation functions. In this letter we introduce a new integrable model whose low-energy behaviour is described by *strongly interacting Luttinger liquid*. We propose that this type of state can account for the observations of high-resolution photoemission experiments on the two- and one-dimensional compounds $K_{0.3}MoO_3$ and $(TaSe_4)_2I$ which show extremely low spectral intensity at the Fermi level. Hence the density distribution function has no peculiarities at the Fermi level [2]. In both systems such behaviour is realized above the Peierls temperature, where strong fluctuations modify the Fermi-step behaviour.

We shall show that such a behaviour of electrons is found in the model studied below in the high-electron-density region (with strong density–density fluctuations) near the metal–insulator phase transition. This suggests that long-range strong repulsion is present in these materials and may account for the observations. We propose models incorporating an integrable form of long-range repulsion. The particular form, though not realistic, allows a complete analysis of the long-range properties of the model, which we expect to be in the same universality describing these materials.

0953-8984/01/440891+07$30.00  

We study an integrable version of the degenerate SU($\nu$) $t$–$J$ model with a hard-core repulsion forbidding electrons at distances less than or equal to $\Delta$, measured in units of the lattice spacing. We shall show that its low-energy behaviour is governed by a strongly interacting Luttinger liquid characterized by a large value of the critical exponent for the momentum distribution function. This state appears at high electron density for any $\Delta \geqslant 1$. Roughly speaking, this electron state is a mirror image of the free Luttinger-liquid state [3] that governs the supersymmetric $t$–$J$ model with long-range interactions [4]. There is a well defined Fermi surface in the free Luttinger-liquid state; a residual Fermi surface remains in the Luttinger-liquid state but it has disappeared in the strongly interacting Luttinger-liquid state.

The dynamics of the hard-core fermions is described by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{L} \left\{ \mathcal{P}_\Delta \left[ -t \sum_{\beta=1}^{\nu} (c^\dagger_{i+1,\beta} c_{i,\beta} + c^\dagger_{i,\beta} c_{i+1,\beta}) + J \sum_{\beta,\gamma}^{\nu} (c^\dagger_{i,\beta} c_{i,\gamma} c^\dagger_{i+1+\Delta,\gamma} c_{i+1+\Delta,\beta} - n_{i,\beta} n_{i+1+\Delta,\gamma}) \right] \mathcal{P}_\Delta \right\} \tag{1}$$

where $c^\dagger_{i,\beta}$ and $c_{i,\beta}$ are the creation and annihilation operators of fermions with colour index $\beta = 1, 2, \ldots, \nu$ at lattice site $i$, $t$ is the hopping integral, $J$ is the constant of the exchange interaction, $\mathcal{P}_\Delta$ is the projector forbidding there being two particles at distances less than or equal to $\Delta$. The case $\Delta = 0$ recovers the projector operator, when the occurrence of two electrons on the same lattice site is forbidden. By $n_{i,\beta} = c^\dagger_{i,\beta} c_{i,\beta}$ we denote the number operator for electrons on site $i$ with colour index $\beta$. The system consists of $N$ electrons on the chain with $L$ sites ($L$ is assumed to be even).

We now turn to the diagonalization of the model Hamiltonian (1) using the coordinate Bethe-*ansatz* approach. The Schrödinger wave function takes the form

$$\psi(x_1, x_2, \ldots, x_N)_{\beta_1,\beta_2,\ldots,\beta_N} = \sum_P \operatorname{sgn}(P) A(P|Q)_{\beta_1,\beta_2,\ldots,\beta_N} \exp\left( \mathrm{i} \sum_{j=1}^{N} k_{Pj} x_{Qj} \right) \tag{2}$$

where the $P$-summation extends over all the permutations of the momenta $\{k_j\}$ and $Q = \{Q_1, \ldots, Q_N\}$ is the permutation of the $N$ particles such that their coordinates satisfy $1 \leqslant x_{Q1} \leqslant x_{Q2} \leqslant \cdots \leqslant x_{QN} \leqslant L$. The coefficients $A(P|Q)$ arising from the different permutations $Q$ are connected by the two-particle $S$-matrix:

$$S(k_i, k_j) = \exp[-\mathrm{i}\Delta(k_i - k_j)] \frac{\lambda_i - \lambda_j - \mathrm{i}P_{ij}}{\lambda_i - \lambda_j - \mathrm{i}} \tag{3}$$

where the operator $P_{ij}$ interchanges the colour indices of the particles and the $\lambda_j$ are the 'dressed' momenta rapidities that are related to the momenta by the following relations:

$$\lambda_j = \begin{cases} -\dfrac{1}{2} \cot \dfrac{k_j}{2} & \text{for } J = t \\ \dfrac{1}{2} \tan \dfrac{k_j}{2} & \text{for } J = -t. \end{cases}$$

The $S$-matrix follows from studying Schrödinger's equation for two particles, when they occupy the 'interacting' lattice sites at distances $\Delta + 1$. Though the resulting $S$-matrix satisfies the Yang–Baxter equation, integrability is not guaranteed on the lattice since three particles (or more) may interact at a time and destroy it. A direct calculation shows that this is not the case here and the model is integrable.

The problem of diagonalizing the colour degrees of freedom encoded in the $S$-matrix (3) can be solved by standard algebraic methods. For a state whose symmetry is specified by a Young tableau with $\nu$ rows of length $n_i, i = 1, \dots, \nu$, we introduce the colour rapidities $\lambda_\alpha^{(r)}$ ($\alpha = 1, 2, \dots, M_r; r = 0, \dots, \nu - 1$), where $M_r = \sum_{1+r}^{\nu} n_i$ is the number of rapidities in the set $\{\lambda_\alpha^{(r)}\}$, $M_\nu = 0$, $M_0 = N$. The colour rapidities satisfy the following nested Bethe-*ansatz* equations:

$$\left(\frac{\lambda_j^{(0)} - \mathrm{i}/2}{\lambda_j^{(0)} + \mathrm{i}/2}\right)^{L-\Delta N} = \prod_{i=1}^{N}\left(\frac{\lambda_i^{(0)} + \mathrm{i}/2}{\lambda_i^{(0)} - \mathrm{i}/2}\right)^{\Delta} \prod_{\alpha=1}^{M_1} \frac{\lambda_j^{(0)} - \lambda_\alpha^{(1)} - \mathrm{i}/2}{\lambda_j^{(0)} - \lambda_\alpha^{(1)} + \mathrm{i}/2}$$

$$\prod_{j=1}^{N} \frac{\lambda_\alpha^{(1)} - \lambda_j^{(0)} + \mathrm{i}/2}{\lambda_\alpha^{(1)} - \lambda_j^{(0)} - \mathrm{i}/2} \prod_{\delta=1}^{M_2} \frac{\lambda_\alpha^{(1)} - \lambda_\delta^{(2)} + \mathrm{i}/2}{\mu_\alpha^{(1)} - \lambda_\delta^{(2)} - \mathrm{i}/2} = -\prod_{\beta=1}^{M_1} \frac{\lambda_\alpha^{(1)} - \lambda_\beta^{(1)} + \mathrm{i}}{\lambda_\alpha^{(1)} - \lambda_\beta^{(1)} - \mathrm{i}} \tag{4}$$

$$\prod_{\gamma=1}^{M_{r-1}} \frac{\lambda_\alpha^{(r)} - \lambda_\gamma^{(r-1)} + \mathrm{i}/2}{\lambda_\alpha^{(r)} - \lambda_\gamma^{(r-1)} - \mathrm{i}/2} \prod_{\delta=1}^{M_{r+1}} \frac{\lambda_\alpha^{(r)} - \lambda_\delta^{(r+1)} + \mathrm{i}/2}{\lambda_\alpha^{(r)} - \lambda_\delta^{(r+1)} - \mathrm{i}/2} = -\prod_{\beta=1}^{M_r} \frac{\lambda_\alpha^{(r)} - \lambda_\beta^{(r)} + \mathrm{i}}{\lambda_\alpha^{(r)} - \lambda_\beta^{(r)} - \mathrm{i}}$$

$$\text{for } r = 2, \dots, \nu - 1; \alpha = 1, \dots, M_r$$

and, in terms of the rapidities $\lambda_j^{(r)}$, the eigenvalues and the magnetization are given by

$$E = -2JN + J\sum_{j=1}^{N} \frac{1}{(\lambda_j^{(0)})^2 + \frac{1}{4}} \tag{5}$$

$$S^z = \frac{1}{2}(\nu - 1)N - \sum_{r=1}^{\nu-1} M_r. \tag{6}$$

The structures of the Bethe-*ansatz* equations are independent of the sign of $J$, but the ground state and the excitations above it depend on it. We will briefly summarize the results of the exact solution of the model for the antiferromagnetic coupling $J = 1$. More detailed discussion of the model and its anisotropic variant will be given elsewhere.

In the thermodynamic limit the rapidities have, in general, complex values:

(i) Real charge rapidities, belonging to the set $\lambda_j^{(0)}$, and corresponding to unpaired electrons.
(ii) Strings of complex spin rapidities, representing colour states

$$\lambda_{\alpha,n,k}^{(r)} = \lambda_{\alpha,n}^{(r)} + \mathrm{i}(n - 2k + 1)/2 \qquad \text{for } k = 1, 2, \dots, n.$$

(iii) Complex spin and charge rapidities describing bound complexes of $m$ electrons ($m \leqslant \nu$).

The ground state consists of bound complexes (for a discussion of the validity of the string hypothesis involving complex momenta, see reference [5]):

$$\lambda_p^{(r)} = \Lambda + \mathrm{i}p/2 \qquad \text{for } r \leqslant \nu - 1;\ p = -(\nu - r - 1), -(\nu - r - 3), \dots, (\nu - r - 1).$$

In the ground state the Bethe equations (4) reduce to sets of coupled linear integral equations for the 'particle' $\rho(\Lambda)$, $\sigma_m^{(r)}(\lambda)$, and 'hole' $\rho_h(\Lambda)$, $\sigma_{hm}^{(r)}(\lambda)$, density functions. After Fourier transforming these equations we obtain

$$\rho_h^{(r)}(\omega) + \rho^{(r)}(\omega) + \sum_{q=1}^{\infty} \sigma_q^{(r+1)} \exp(-q|\omega|/2$$
$$+ \sum_{l=0}^{\nu-1} \rho^{(l)}(\omega) \exp[-(r + l - Q_{r,l})|\omega|/2] \frac{\sinh[(p_{r,l} + 1)\omega/2]}{\sinh(\omega/2)}$$
$$= (1 - \Delta n) \exp[-(r + 1)|\omega|/2] \qquad \text{for } r = 0, 1, \dots, \nu - 1 \tag{7}$$

$$\sigma_{hm}^{(r)}\omega) + \sum_{n=1}^{\infty}[2\cosh(\omega/2)\sigma_n^{(r)}(\omega) - \sigma_n^{(r+1)}(\omega) - \sigma_n^{(r-1)}(\omega)]$$
$$\times \exp[-\max(m,n)|\omega|/2]\frac{\sinh[\min(m,n)\omega/2]}{\sinh(\omega/2)}$$
$$= \rho^{(r-1)}(\omega)\exp(-m|\omega|/2) \qquad \text{for } r = 1, 2, \ldots, \nu - 1 \tag{8}$$

where $Q_{r,l} = \min(r,l) - \delta_{r,l}$. The last set of equations hold for an arbitrary $m = 1, 2, \ldots, \infty$ with $\sigma_m^{(0)}(\lambda), \sigma_{hm}^{(0)}(\lambda), \sigma_m^{(\nu)}(\lambda), \sigma_{hm}^{(\nu)}(\lambda)$ being identically zero. Apart from the driving terms, these equations are identical to those of the degenerate supersymmetric $t$–$J$ model [6] and the degenerate electron gas with an attractive $\delta$-function potential [7].

Expressing the ground-state energy density, $\varepsilon = E/L$, in terms of the solution densities we obtain

$$\varepsilon = -2n + \sum_{r=0}^{\nu-1}\int \mathrm{d}\Lambda\ \rho^{(r)}(\Lambda)\frac{r+1}{\Lambda^2 + (r+1)^2/4} \tag{9}$$

where the total density of electrons is given by

$$n = \sum_{l=0}^{\nu-1}(l+1)\int \mathrm{d}\Lambda\ \rho^{(l)}(\Lambda). \tag{10}$$

Using the Bethe-*ansatz* equation we study the Fermi velocity of electrons and long-distance power-law behaviour of the correlation functions in the ground state. The analysis of low-lying excitations shows that there are one charge and $\nu - 1$ spin gapless excitations [6]. The Fermi velocity of the charge gapless excitation $v_c$ is given by $v_c = |\epsilon'(Q)|/[2\pi\rho^{(\nu-1)}(Q)]$ (the prime denotes a derivative). The dressed energy $\epsilon(\Lambda)$ is the solution of the following integral equation:

$$\epsilon(\Lambda) - \int_{-Q}^{Q}\mathrm{d}\Lambda'\ K_{\nu-1}(\Lambda - \Lambda')\epsilon(\Lambda') = -2\pi K_1(\Lambda) - \mu' \tag{11}$$

and the kernel $K_j(\Lambda)$ is

$$K_j(\Lambda) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\ \frac{\sinh(\omega j/2)}{\sinh(\omega\nu/2)}\exp(\mathrm{i}\omega\Lambda - |\omega|/2) \tag{12}$$

and also the $\Lambda$ Fermi level $Q$ is defined by $\epsilon(\pm Q) = 0$, where $\mu'$ is the chemical potential.

The Bethe-*ansatz* equations (7), (8) can be solved numerically for arbitrary values of the parameter $\Delta$ and the electron density. Numerical results for the ground-state energy per lattice site are presented in figure 1 for several values of the parameter $\Delta$. For the sake of comparison we have presented the ground-state energy of the degenerate (dotted line) $t$–$J$ model [6]. We clearly observe that the energy increases with $\Delta$. The minimum value is realized near or at the extreme density $n_{max} = 1/(1+\Delta)$, depending on $\Delta$ and $\nu$.

In figure 2 we show the Fermi velocity obtained numerically for several values of $\Delta$ and $\nu = 3$. The system is metallic except for $n \to 0, n_{max}$ where the Fermi velocity tends to zero. The density $n_{max}$ corresponds to a fully filled electron subband when the dynamics of the electrons is frozen, and a Mott transition to an insulating phase occurs—to a Heisenberg system with a new spacing parameter $\Delta + 1$. Note that the height of the maximum shifts towards the high-electron-density region and increases with $\Delta$. The value of $v_c$ decreases with $\nu$.

The long-distance behaviour of the charge-density $n(r)$ correlator is characterized by the exponents $\eta_j$ [8] ($k_F = \pi n$ is the Fermi momentum):

$$\langle n(r)n(0)\rangle \simeq n^2 + A_0 r^{-2} + \sum_{j=1}^{\nu} A_j\cos(2jk_F r)r^{-\eta_j}.$$

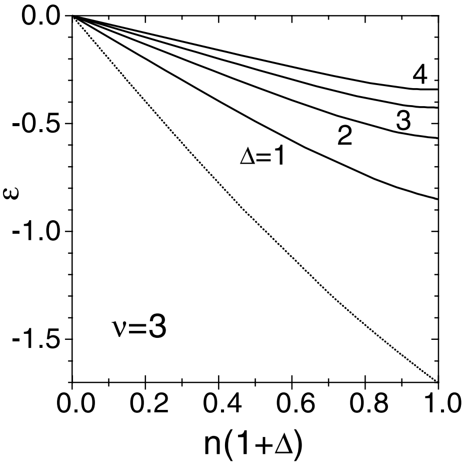


**Figure 1.** The ground-state energy per site versus the electron density. The dotted line represents the case $\Delta = 0$. The individual curves are labelled by the values of the parameter $\Delta$.

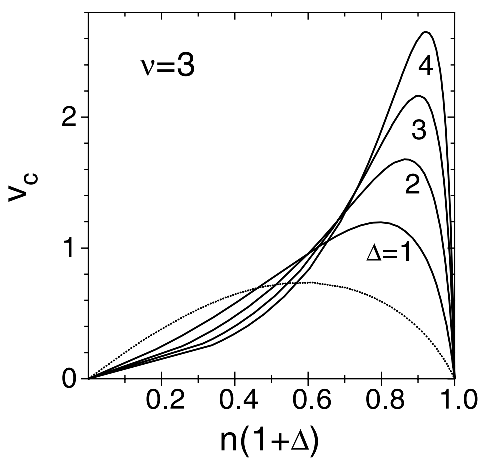


**Figure 2.** The Fermi velocity—similar to that for figure 1.

The exponents can be expressed as

$$\eta_j = \frac{2j(\nu - j)}{\nu} + \frac{j^2\alpha}{\nu^2}. \tag{13}$$

Here $\alpha = 2[\xi(Q)]^2$, with $\xi(Q)$ the dressed charge at the $\Lambda$ Fermi surface, the dressed charge function $\xi(\Lambda)$ being defined through the integral equation

$$\xi(\Lambda) - \int_{-Q}^{Q} \mathrm{d}\Lambda'\, K_{\nu-1}(\Lambda - \Lambda')\xi(\Lambda') = 1 - \Delta n. \tag{14}$$

The exponent $\alpha$ is plotted in figure 3. We observe that it depends on the electron density, its value decreasing monotonically from $2\nu$ to $2/(1+\Delta)^2$ as $n$ increases from zero to $n_{max}$.

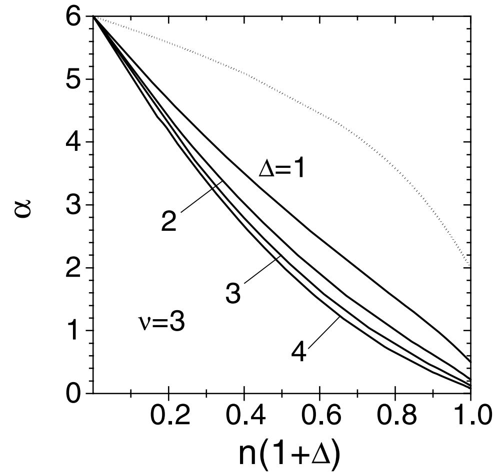


**Figure 3.** The exponent $\alpha$ as a function of the electron density.

The momentum distribution function close to $k_F$ is determined by the exponent $\Theta$:

$$\langle n_k \rangle \simeq \langle n_{k_F} \rangle - \text{constant} \times |k - k_F|^{\Theta} \operatorname{sgn}(k - k_F) \tag{15}$$

where the exponent $\Theta$ is given by

$$\Theta = \frac{1}{\alpha}\left(1 - \frac{\alpha}{2\nu}\right)^2. \tag{16}$$

$\Theta$ increases monotonically from zero to $\frac{1}{2}(1+\Delta-1/[\nu(1+\Delta)])^2$ with the density. For $\Delta \geqslant 1$ there will be some density $n_c$ for which $\Theta(n_c) = 1$. In the low-density limit the critical exponents $\eta_1 = 2$ and $\Theta = 0$ are the canonical exponents characteristic of the noninteracting electron system.

In figure 4 we plot $\Theta$ as a function of the density for $\nu = 3$, and indicate the values of $n_c$ where there is a crossover from a Luttinger-liquid regime ($0 < \Theta < 1$) to a *strongly interacting Luttinger-liquid regime* ($\Theta > 1$), where the residual Fermi surface has disappeared. The density $n_c$ decreases with the increase of $\Delta$ and the region of strongly interacting Luttinger liquid grows. In this regime the hard-core potential dominates. The behaviour in this regime resembles the $\nu = 1$ case which does not include the exchange interaction and describes spinless fermions interacting only via hard-core repulsion.

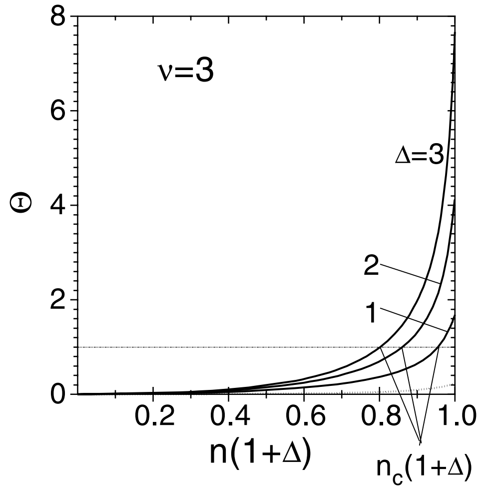


**Figure 4.** $\Theta$ characterizing the Fermi-point singularity of the momentum distribution function as a function of the electron density for $\nu = 3$, $\Delta = 0$ (dotted line) and $\Delta = 1, 2, 3$ (solid lines). The broken line separates off the strongly interacting Luttinger-liquid state.

In summary, we have presented a soluble generalization of the multicomponent $t$–$J$ model, leading to nontrivial Luttinger-liquid behaviour. We have obtained the exact correlation exponents for an arbitrary density of the electrons. The limit $\Theta \to 0$ that corresponds to the noninteracting electron system is realized in the low-density limit. A radically different situation is found for high electron density, where a strongly interacting Luttinger liquid with $\Theta > 1$ appears. This state is due to the strong competition between the exchange interaction and the hard-core repulsion potential that dominates at high electron density. In the extreme high-density limit the system undergoes a metal–insulator transition, where the insulator is described by an isotropic Heisenberg chain with a spacing parameter $\Delta + 1$.

IK wishes to thank for support the Visitor Programme of the Max-Planck-Institut für Physik Komplexer Systeme, Dresden, Germany. This work was partially done within the framework of the Associate Membership programme of the International Centre for Theoretical Physics, Trieste, Italy.

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